

FIBONACCI NUMBERS AND IDENTITIES II

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1. INTRODUCTION

Let $a, b, p, q \in \mathbb{C}$, $q \neq 0$. Define the generalised Fibonacci sequence $\{W_n\} = \{W_n(a, b; p, q)\}$ by $W_0 = a$, $W_1 = b$,

$$W_n = pW_{n-1} - qW_{n-2}. \quad (1.1)$$

Obviously the definition can be extended to negative subscripts ; that is, for $n = 1, 2, 3, \dots$, define

$$W_{-n} = (pW_{-n+1} - W_{-n+2})/q. \quad (1.2)$$

In the case $a = 0$, $b = 1$, following [Ho], we shall denote the sequence $\{W_n(0, 1; p, q)\}$ by $\{u_n\}$. Equivalently,

$$u_n = W_n(0, 1; p, q). \quad (1.3)$$

Note that $u_{-n} = -q^{-n}u_n$. In this article, we will give a unified proof of various identities involving W_n . The alternative proof presented in this article (for Melham's, Howard's and Horadam's identities) uses nothing but *recurrence* which is slightly different from the original proof.

2. RECURRENCE RELATION

Lemma 2.1. $A(n) = W_{2n}$, $B(n) = W_n W_{n+r}$ and $C(n) = q^n$ satisfy the following recurrence relation

$$X(n+3) = (p^2 - q)X(n+2) + (q^2 - p^2q)X(n+1) + q^3X(n). \quad (2.1)$$

Proof. One sees easily that q^n satisfies (2.1). The following shows that W_n^2 satisfies (2.1). By (1.1),

$$W_{n+3}^2 = (pW_{n+2} - qW_{n+1})^2 = p^2W_{n+2}^2 + q^2W_{n+1}^2 - 2pqW_{n+2}W_{n+1}, \quad (2.2)$$

$$2pqW_{n+2}W_{n+1} = qW_{n+2}(W_{n+2} + qW_n) + pq(pW_{n+1} - qW_n)W_{n+1}. \quad (2.3)$$

Replace the last quantity of the right hand side of (2.2) by (2.3), one has the following.

$$W_{n+3}^2 = (p^2 - q)W_{n+2}^2 + (q^2 - p^2q)W_{n+1}^2 - q^2W_{n+2}W_n + pq^2W_{n+1}W_n. \quad (2.4)$$

By (1.1), $q^2W_{n+2}W_n - pq^2W_{n+1}W_n = -q^3W_n$. This completes the proof of the fact that W_n^2 satisfies the recurrence relation (2.1). One can show similar to the above that W_{2n} and $W_n W_{n+r}$ satisfy the recurrence (2.1). \square

Lemma 2.2. Suppose that $x(n)$ and $y(n)$ satisfy the recurrence (2.1). Let $r, s \in \mathbb{Z}$. Then $x(n) \pm y(n)$ and $rx(n+s)$ satisfy the recurrence (2.1).

Lemma 2.3. Suppose that both $A(n)$ and $B(n)$ satisfy (2.1). Then $A(n) = B(n)$ if and only if $A(n) = B(n)$ for $n = 0, 1, 2$.

Proof. Since both $A(n)$ and $B(n)$ satisfy the same recurrence relation, $A(n) = B(n)$ if and only if they satisfy the same initial condition. This completes the proof of the lemma. \square

Lemma 2.4. *Let W_n and u_n be given as in (1.1) and (1.3). Then the following hold.*

$$W_r = u_r W_1 - q u_{r-1} W_0, \quad (2.5)$$

$$W_r = u_{r-1} W_2 - q u_{r-2} W_1, \quad (2.6)$$

$$W_r = u_{r-2} W_3 - q u_{r-3} W_2. \quad (2.7)$$

Proof. We note first that (1.1) and (1.2) give the same recurrence relation. Denoted by $A(r)$ and $B(r)$ the left and right hand side of (2.5)-(2.7). Since W_r , u_r , u_{r-1} and u_{r-2} (as functions in r) satisfy the recurrence relation (1.1), both $A(r)$ and $B(r)$ satisfy (1.1). Hence $A(r) = B(r)$ if and only if they satisfy the same initial condition. (2.5)-(2.7) can now be verified with ease. \square

3. ON MELHAM'S IDENTITY

Theorem 3.1. (Melham [M]) *Let W_n be given as in (1.1) and let $e = pab - qa^2 - b^2$. Then*

$$W_{n+1}W_{n+2}W_{n+6} - W_{n+3}^3 = eq^{n+1}(p^3W_{n+2} - q^2W_{n+1}). \quad (3.1)$$

In the case $p = 1, q = -1, a = b = 1$, (3.1) becomes

$$F_{n+1}F_{n+2}F_{n+6} - F_{n+3}^3 = (-1)^n F_n, \quad (3.2)$$

where F_n is the n -th Fibonacci number. In this short note, we give a direct proof of Theorem 3.1 which uses only the simple fact that $W_n W_{n+r}$ and q^n satisfy the recurrence relation (2.1).

Lemma 3.2. $W_{n+2}W_{n+4} - W_{n+3}^2 = eq^{n+2}$ and $W_{n+1}W_{n+6} - W_{n+3}W_{n+4} = eq^{n+1}(p^3 - pq)$.

Proof. Let $A(n) = W_{n+2}W_{n+4} - W_{n+3}^2$, $B(n) = eq^{n+2}$. By Lemmas 2.1 and 2.2, both $A(n)$ and $B(n)$ satisfy the recurrence relation (2.1). Note that $A(n) = B(n)$ for $n = 0, 1, 2$. It follows from Lemma 2.3 that $A(n) = B(n)$ for all n . The second identity can be proved similarly. \square

Proof of Theorem 3.1. By the first identity of Lemma 3.2, the left hand side of (3.1) can be written as

$$W_{n+1}W_{n+2}W_{n+6} - W_{n+3}^3 = W_{n+2}(W_{n+1}W_{n+6} - W_{n+3}W_{n+4}) + eq^{n+2}W_{n+3}. \quad (3.3)$$

Replace the left hand side of (3.1) by identity (3.3), one has (3.1) is true if and only if

$$W_{n+2}(W_{n+1}W_{n+6} - W_{n+3}W_{n+4}) + eq^{n+2}W_{n+3} = eq^{n+1}(p^3W_{n+2} - q^2W_{n+1}). \quad (3.4)$$

By (1.1), $W_{n+3} = pW_{n+2} - qW_{n+1}$ (note that (1.1) and (1.2) give the same recurrence relation). Hence (3.4) is equivalent to

$$W_{n+2}(W_{n+1}W_{n+6} - W_{n+3}W_{n+4}) = eq^{n+1}(p^3 - pq)W_{n+2}. \quad (3.5)$$

One may now apply the second identity of Lemma 3.2 to finish our proof of Theorem 3.1. \square

4. ON HOWARD'S IDENTITY

Let p and q be given as in (1.1). Define similarly the sequence $\{V_n\}$ by the same recurrence relation

$$V_n = pV_{n-1} - qV_{n-2}, \quad V_0 = c, \quad V_1 = d. \quad (4.1)$$

A careful study of the proof of Lemma 2.1 reveals that

- (i) the fact we used to show that W_n^2 satisfies the recurrence (2.1) is the recurrence (1.1),
- (ii) the proof of Lemma 2.1 is independent of the choice of the values of W_0 and W_1 .

As a consequence, the following lemma is clear.

Lemma 4.1. *Let $k, r \in \mathbb{Z}$. Then rW_nV_{n+k} , rW_nu_{n+k} and rV_nu_{n+k} satisfy the recurrence relation (2.1).*

Theorem 4.2. (Howard [Ho]) *Let W_n, V_n and u_n be given as in (1.1), (1.3) and (4.1). Then*

$$W_n^2 - q^{n-j}W_j^2 = u_{n-j}(bW_{n+j} - qaW_{n+j-1}), \quad (4.2)$$

$$W_{m+n+1} = W_{m+1}u_{n+1} - qW_mu_n, \quad (4.3)$$

$$V_{m+k}W_{n+k} - q^kV_mW_n = u_k(bV_{m+n+k} - qaV_{m+n+k-1}), \quad (4.4)$$

$$W_{n+k}^2 - q^{2k}W_{n-k}^2 = u_{2k}(bW_{2n} - qaW_{2n-1}). \quad (4.5)$$

In [Ho], proof of Theorem 4.2 involves mathematical induction, a result of Bruckman [B] and a very neat combinatorial argument. In this article, we give an alternative proof which is direct and uses Lemmas 2.1-2.4 and 4.1 only.

Proof of Theorem 4.2. We shall prove (4.3). The rest can be proved similarly. Note first that (4.3) can be proved by induction as well. Let $m+n+1=r$. Since $u_{-n} = -q^{-n}u_n$ and $q \neq 0$, (4.3) becomes

$$-q^{m-r}W_r = W_{m+1}u_{m-r} - W_mu_{m+1-r}. \quad (4.6)$$

Let $A(m) = -q^{m-r}W_r$ and $B(m) = W_{m+1}u_{m-r} - W_mu_{m+1-r}$. View both $A(m)$ and $B(m)$ as functions in m . By Lemmas 2.1 2.2 and 4.1, they both satisfy the recurrence relation (2.1). By Lemma 2.3, $A(m) = B(m)$ if and only if $A(m) = B(m)$ for $m = 0, 1$ and 2 . Equivalently, (4.3) is true if and only if $A(0) = B(0)$, $A(1) = B(1)$ and $A(2) = B(2)$. Equivalently,

$$-q^{-r}W_r = W_1u_{-r} - W_0u_{1-r}, \quad (4.7)$$

$$-q^{1-r}W_r = W_2u_{1-r} - W_1u_{2-r}, \quad (4.8)$$

$$-q^{2-r}W_r = W_3u_{2-r} - W_2u_{3-r}. \quad (4.9)$$

Since $u_{-n} = -q^{-n}u_n$ and $q \neq 0$, (4.7), (4.8) and (4.9) of the above can be written as the following.

$$W_r = u_rW_1 - qu_{r-1}W_0, \quad (4.10)$$

$$W_r = u_{r-1}W_2 - qu_{r-2}W_1, \quad (4.11)$$

$$W_r = u_{r-2}W_3 - qu_{r-3}W_2. \quad (4.12)$$

Hence (4.3) is true if and only if (4.10)-(4.12) are true. Applying Lemma 2.4, we conclude that (4.3) is true. \square

5. HORADAM'S IDENTITIES

Horadam [H] studied the fundamental arithmetical properties of $\{W_n\}$ and provided us with many interesting identities. A careful checking of his identities suggested that Identities (3.14), (3.15), (4.1)-(4.12) and (4.17)-(4.22) of [H] can be verified by our method as the functions involved in those identities satisfy the recurrence (2.1) of the present paper.

6. DISCUSSION

The technique we presented in the proof of Theorems 3.1 and 4.2 can be applied to the verification of identities of the form $A(n) = B(n)$ if one can show

- (i) $A(n)$ and $B(n)$ satisfy the same recurrence relation,
- (ii) $A(n)$ and $B(n)$ satisfy the same initial condition.

The verification of (i) is not difficult (see Appendix B of [LL]). (ii) can be checked with ease. Note that it is not just our purpose to reprove the identities but to illustrate the importance and usefulness of the *recurrence* (such as (2.1) of the present paper)

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